FUZZY SUBALGEBRAS AND CONGRUENCES

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By

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Synopsis

The introduction of the concept of a fuzzy subset of a nonempty set X as a function of X into the unit interval $[0, 1]$ by L.Zadeh initiated several algebraists to take up the study of fuzzy subalgebras of various algebraic systems such as groups, rings, modules, lattices etc. Rosenfeld defined the notion of a fuzzy subgroup of a group and since then several researchers worked on fuzzy subrings and ideals of rings (Malik and Mordeson), fuzzy ideals of lattices (Attallah, Lehmke and U.M.Swamy and D.V.Raju), fuzzy semi prime ideals of a semi ring (Dutta and Biswas), fuzzy maximal ideals of Gamma near-rings (Jun, Kim and Oztirk), fuzzy prime ideals of lattices and hyper lattices (Koguep, N kuimi and Lele), fuzzy prime ideals of rings (U.M.Swamy and K.L.N.Swamy), algebraic fuzzy systems and irreducibility (U.M.Swamy and D.V.Raju), fuzzy ideals of a ring (Mukharjee and sen), fuzzy groups and level subgroups (Das), fuzzy pseudo ideals in semi groups (Dutta), fuzzy vector spaces and fuzzy topological vector spaces (Katsaras and Liu), etc.

In most of the works mentioned above, the fuzzy statements take truth values in the interval [0,1] of real numbers, while the conventional (or crisp) statements take truth values in the two-element set $\{F, T\}$ or $\{0, 1\}$, where F and 0 stand for 'false' and T or 1 stand for 'true'. However, Gougen realized that the unit interval [0, 1] is insufficient to have the truth values of certain fuzzy statements.

In most of the results in this thesis, we consider algebras with at least one nullary fundamental operation. This is the case with almost all the algebras we are familiar with, like groups, rings, modules, vector space etc. For any algebra $\mathbf{A}$, the underlying set A is called the universe of A and a nonempty subset B of A is called a subalgebra (or subuniverse) of $\mathbf{A}$ if the following are satisfied.

(1) $f^A \in B$ for all nullary of $f \in \mathscr{F}$ and

(2) for any n-ary $f \in \mathscr{F}$, $n > 0$, and $b_1, b_2, ..., b_n \in B$, $f^A(b_1, b_2, ..., b_n) \in B$.

The Thesis is broadly divided into four chapters $0, 1, 2,$ and 3. Chapter 0 is devoted to collect all the necessary preliminaries which will be useful in our discussions in the main text of the thesis. Even though these preliminaries are well known for those working in Universal Algebra and/or Lattice Theory, it will be convenient for others to have all these elementary notions and results in the beginning of the thesis for the sake of ready reference and completeness of the thesis. The proofs of most of the results presented in chapter 0 are either straight forward verifications or well known and hence we simply state the results and skip the proofs.

The main text of the thesis is in chapters $1, 2$ and 3. Chapter 1 is on general fuzzy subalgebras and is subdivided into three sections. In section 1.1, we discuss on distributivities in lattice. As mentioned above, the interval $[0, 1]$ is insufficient to contain the truth values of general fuzzy statements and it is necessary to consider a more general class of lattices in place of $[0, 1]$. U.M. Swamy and others used a complete lattice satisfying the infinite meet distributivity, which are called frames, to have the truth values of general fuzzy statements. For this reason, we make a thorough discussion on various types of distributivities in lattices in section 1.1.

Section 1.2 is devoted to a study L-fuzzy subsets of an arbitrary set, where L is an arbitrary frame. In section 1.3, we consider an (universal) algebra $\mathbf{A}$ and introduce the notion of an L- fuzzy subalgebra of $\mathbf{A}$ and discuss certain general properties of these, by observing that these form an algebraic fuzzy set system.

Chapter 2 is on prime and maximal fuzzy subalgebras of an algebra A and is subdivided into six sections. In section 2.1, we discuss irreducible and prime elements in general lattices in order to investigate these in the lattices of (crisp) subalgebras and fuzzy subalgebras. Section 2.2 is devoted to a study of certain general properties of prime and maximal subalgebras of a given algebra.

In section 2.3, we introduce the notion of a prime L-fuzzy subalgebra of an algebra A as simply a prime element in the lattice of L-fuzzy subalgebras; that is, an L-fuzzy subalgebra P of A is called prime if P is not the constant map $\overline{1}$ and, for any L-fuzzy subalgebras F and G of A,

$$F \wedge G \leq P \Rightarrow \text{either } F \leq P \text{ or } G \leq P.$$

Here, we have mainly proved a characterization theorem for prime L-fuzzy subalgebras which states that an L-fuzzy subalgebra P of A is prime if and only if there is a prime subalgebra B of A and a prime element α in L such that

$$P(a) = \begin{cases} 1 & \text{if } a \in B \\ \alpha & \text{if } a \notin B. \end{cases}$$

This proves that there is a one-to-one correspondence between prime L-fuzzy subalgebra of A and the pairs (B, α) where B is a prime subalgebra of A and α is

a prime element in the frame L.

In section 2.4, we discuss the L-fuzzy subsets F of an algebra A for which each α-cut is either A or a prime subalgebra of A and there are termed as L-fuzzy prime subalgebras. we have proved that every prime L-fuzzy subalgebra is an L-fuzzy prime subalgebra and that the converse is not true, by providing counter examples. Also we have proved that a proper L-fuzzy subalgebra P of A is an L-fuzzy prime subalgebra if and only if $P(A)$ is a chain in L and, for any elements a and b in A,

$$P(a) \vee P(b) = Inf\{P(x) \mid x \in < a > \cap < b >\}$$

where $< a >$ and $< b >$ are subalgebras of A generated by a and b respectively. Further, we have proved that the 1-cut of an L-fuzzy prime subalgebra of A is a prime subalgebra and that the converse is not true, by exhibiting a countable example.

In section 2.5, we discuss maximal L-fuzzy subalgebras of an algebra A which are precisely maximal members in the set of proper L-fuzzy subalgebras of A or, equivalently, the dual atoms in the lattice $\mathscr{FS}_L(A)$ of L-fuzzy subalgebras of A. We have proved that F is a maximal L-fuzzy subalgebra of A if and only if there exist a maximal subalgebra M of A and a dual atom α in L such that

$$F(a) = \begin{cases} 1 & \text{if } a \in M \\ \alpha & \text{if } a \notin M. \end{cases}$$

In such a case, F is denoted by α_M.

This yields a one-to-one correspondence between maximal L-fuzzy subalgebras of A and pairs (M, α), where M is a maximal subalgebra of A and α is a

dual atom in L. When L is no dual atoms (for example, $[0,1]^n$ is one such, for any positive integer n), then any algebra A has no maximal L-fuzzy subalgebras.

Finally, in section 2.6, we introduce the notion of an L-fuzzy maximal subalgebra as a proper L-fuzzy subalgebra each of whose α-cuts are either a maximal subalgebra of A or the whole of A. We have proved that F is an L- fuzzy maximal subalgebra if and only if $F = \alpha_M$ for some maximal subalgebra M of A and $\alpha < 1$ in L. From this, we have derived that every maximal L-fuzzy subalgebra of A is an L-fuzzy maximal subalgebra. Counter examples are provided to disprove the converse.

Chapter 3 is on L-fuzzy congruences. For any algebra A of type $\mathscr{F}$, an equivalence relation on A is called a congruence if it is compatible with all the fundamental operations on A. It is known that congruences on a group (ring or R-module) are in one-to-one correspondence with the normal subgroups (ideals or R-submodules respectively). Further congruences on any algebra A of type $\mathscr{F}$ are precisely same as subalgebras the augmented algebra $A \times A$ of $\mathscr{F}'$, where $\mathscr{F}'$ is $\mathscr{F}$ together a nullary operation corresponding to each $(a,a) \in A \times A$, a unary operation s defined by $s(a,b) = (b,a)$ for all $(a,b) \in A \times A$ and a binary operation t on $A \times A$ defined by

$$t((a,b),(c,d)) = \begin{cases} (a,d) & \text{if } b = c \\ (a,b) & \text{if } b \neq c \end{cases}$$

with this in view, congruences on algebra A can be treated as subalgebras of some algebraic structure on $A \times A$ and hence all the concepts and results on subalgebras or L-fuzzy subalgebras can be easily extended to those on congruences and L-fuzzy congruences. In section 3.1, we formally define the notion of an L-fuzzy congruence

on an algebra and state certain equivalent formulations. In section 3.2, we introduce the notions of prime L-fuzzy congruences and L-fuzzy prime congruences and state the results on these analogous to those of L-fuzzy subalgebras in sections 2.3 and 2.4. Since the proofs of these are analogous to those in sections 2.3 and 2.4, we simply state these and skip the proofs. In section 3.3, we briefly discuss the notions and results on maximal L-fuzzy congruences and L-fuzzy maximal congruences on an algebra and simply state the results by skipping the proof, since these are analogous to those corresponding results in sections 2.5 and 2.6.

Research Papers Published/ Communicated:

1. *On the Truth Values of Fuzzy Statements*, International Journal of Applied Computational Science and Mathematics. ISSN 2249-3042 Volume 3, Number 1 (2013) pp. 1-6 (jointly with U.M.Swamy).

2. *Fuzzy prime ideals and filters of lattices* (communicated) (jointly with U.M.Swamy and G.Nanaji Rao)

3. *Fuzzy maximal ideals and filters of lattices* (communicated) (jointly with U.M.Swamy and G.Nanaji Rao)

4. *Prime L-fuzzy subalgebras of an algebra* (communicated) (jointly with U.M.Swamy and G.Nanaji Rao)

5. *Maximality in L-fuzzy subalgebras of an algebra* (communicated) (jointly with U.M.Swamy and G.Nanaji Rao)

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